The fundamental spherically symmetric fluid model

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Abstract

It is shown that an effective anisotropic spherically symmetric fluid model with heat flow can absorb the addition to a perfect fluid of pressure anisotropy, heat flow, bulk and shear viscosity, electric field and null fluid. In most cases the induction of effective heat flow can be avoided. There is a relationship between anisotropic and charged perfect fluids.

PACS numbers: 04.20 Cv, 04.20 Dg, 04.40 Nr

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I.   INTRODUCTION

Spherically symmetric perfect fluid solutions in general relativity have been studied from its very beginning, starting with the interior Schwarzschild solution. Gradually different mechanisms in stellar models have been identified that create pressure anisotropy [1], [2]. Radiating spherical collapse demands the introduction of heat flow [3] or null fluid which describe energy dissipation in different approximations [4]. More realistic fluids possess also bulk and shear viscosity [5], [6], [7]. Charged perfect fluids or dust have been discussed by many authors [8]. It has been shown too that the sum of two perfect fluids, two null fluids or a perfect and a null fluid can be represented by effective anisotropic fluid models [9]. In view of the many existing relations among the fluid models mentioned above it is hard to point out some center.

In this paper it is shown that the anisotropic fluid with heat flow is in some sense the most fundamental model and can absorb the addition of viscosity, charge and null fluids. Something more, heat flow is not generated in most cases.

In Sec.II the basic anisotropic fluid model is defined. In Sec.III the fluid is supplied with bulk and shear viscosity which leads to a new effective anisotropic model. In Sec.IV the same is done for the addition of charge and in Sec.V null fluids are accommodated into the anisotropic model. Sec.VI summarizes all additions and the effective characteristics of the general anisotropic model are given. Several conclusions are drawn.

II.   ANISOTROPIC FLUID MODEL WITH HEAT FLOW

Einstein's field equations are given by

$$8\pi T_{\alpha\beta} = G_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, $T_{\alpha\beta}$ is the energy-momentum tensor (EMT) and units are used so that $c = G = 1$. The general spherically symmetric metric is written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta \ d\varphi^2 \right) \tag{2}$$

where A, B, R are positive functions of t and r only. The spherical coordinates are numbered as $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$. The Einstein tensor involves the Ricci tensor and

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scalar which are given by the metric and its first and second derivatives [4]. Its non-trivial components are $G_{00}, G_{01}, G_{11}, G_{22} = \sin^{-2}\theta \, G_{33}$.

We are interested in the structure of EMT. It reads for anisotropic fluids with heat flow

$$T_{\alpha\beta} = (\mu + p_t) \, u_\alpha u_\beta + p_t g_{\alpha\beta} + (p_r - p_t) \, \chi_\alpha \chi_\beta + q_\alpha u_\beta + u_\alpha q_\beta. \tag{3}$$

Here μ is the energy density, p_r is the radial pressure, p_t is the tangential pressure, u^α is the four velocity of the fluid (a timelike vector), χ^α is a unit spacelike vector along the radial direction and q^α is the heat flux (in the radial direction too). We have

$$u^\alpha u_\alpha = -1, \quad \chi^\alpha \chi_\alpha = 1, \quad u^\alpha \chi_\alpha = 0, \quad u^\alpha q_\alpha = 0. \tag{4}$$

It is assumed that the coordinates are comoving, hence, the fluid is motionless in them

$$u^\alpha = A^{-1}\delta_0^\alpha, \quad \chi^\alpha = B^{-1}\delta_1^\alpha, \quad q^\alpha = qB^{-1}\delta_1^\alpha \tag{5}$$

where $q = q\,(r,t)$. This gives

$$T_{00} = \mu A^2, \quad T_{01} = -qAB, \quad T_{11} = p_r B^2, \quad T_{22} = p_t R^2, \tag{6}$$

which should be plugged in the Einstein equations. The anisotropic fluid does not radiate when $q = 0$ and becomes perfect when $p_r = p_t$. Thus it accommodates anisotropy of pressure and heat flow when one starts with a perfect fluid model.

Let us see now what happens when other EMTs are added to the basic anisotropic one.

III.   BULK AND SHEAR VISCOSITY

Bulk viscosity [5], [6], [7] adds to the basic EMT the following piece

$$T_{\alpha\beta}^B = -\zeta\Theta h_{\alpha\beta} \tag{7}$$

where ζ is a coefficient, Θ is the expansion of the fluid and $h_{\alpha\beta}$ is the projector on the hyperplane orthogonal to u^α

$$\Theta = u^\alpha_{;\alpha}, \quad h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \tag{8}$$

Obviously, this means the appearance of effective pressures

$$p_r^B = p_t^B = -\zeta\Theta, \tag{9}$$

which should be added to p_r, p_t. They do not change the degree of anisotropy $\Delta p = p_r - p_t$. Thus even perfect fluid can absorb the bulk viscosity. The quantities μ, q remain the same. No heat flow is generated in particular.

Shear viscosity is responsible for the piece

$$T_{\alpha\beta}^S = -2\eta h_{\alpha\gamma} h_{\beta\delta} \sigma^{\gamma\delta} \tag{10}$$

where $\sigma_{\alpha\beta}$ is the shearing tensor

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} + a_{(\alpha}u_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}, \tag{11}$$

η is some coefficient and a_α is the acceleration

$$a_\alpha = u_{\alpha;\beta}u^\beta. \tag{12}$$

The shearing tensor satisfies the conditions

$$\sigma_{\alpha\beta}u^\beta = 0, \quad \sigma_{\alpha\beta}g^{\alpha\beta} = 0, \tag{13}$$

hence, Eq (10) transforms into

$$T_{\alpha\beta}^S = -2\eta\sigma_{\alpha\beta}. \tag{14}$$

Use of Eqs (5,11) gives the non-zero components of the shear [4]

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma \tag{15}$$

where

$$\frac{2}{3}\sigma^2 = \sigma^{\alpha\beta}\sigma_{\alpha\beta}. \tag{16}$$

One can check that the same components follow when $\sigma_{\alpha\beta}$ is written as the tensor

$$\sigma_{\alpha\beta} = -\frac{1}{3}\sigma h_{\alpha\beta} + \sigma\chi_\alpha\chi_\beta. \tag{17}$$

Thus it coincides with the general shear tensor defined by Eq (11) in the spherically symmetric case. It also satisfies relations (13) in any metric.

Plugging Eq (17) into Eq (14) and comparing it to Eq (3) we find the effectively generated pressures

$$p_r^S = -2p_t^S = -\frac{4}{3}\eta\sigma. \tag{18}$$

The degree of anisotropy is changed. There is no generation of energy density or heat flow. The scalars Θ, σ can be expressed through the metric and its first derivatives.

IV. ELECTROMAGNETIC FIELDS

The EMT of electromagnetic fields is given by

$$T_{\alpha\beta}^{EM} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\beta}^{\mu} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right) \tag{19}$$

where $F_{\mu\nu}$ is the Faraday tensor. One defines a unit timelike vector field n^{μ}. An observer moving on its direction will measure electric and magnetic field respectively

$$E_{\alpha} = F_{\alpha\mu}n^{\mu}, \quad H_{\alpha} = \frac{1}{2}\varepsilon_{\alpha\mu\nu}F^{\mu\nu}. \tag{20}$$

These fields are spacelike, $E^{\alpha}n_{\alpha} = H^{\alpha}n_{\alpha} = 0$. The Faraday tensor decomposes like [10]

$$F_{\alpha\beta} = \varepsilon_{\alpha\beta\mu}H^{\mu} - 2E_{[\alpha}n_{\beta]}. \tag{21}$$

Plugging this expression into Eq (19) gives formula (7) from Ref [10], which becomes after some rearrangements

$$T_{\alpha\beta}^{EM} = \frac{1}{4\pi}\left(H^2 + E^2\right)\left(n_{\alpha}n_{\beta} + \frac{1}{2}g_{\alpha\beta}\right) - \frac{1}{4\pi}\left(E_{\alpha}E_{\beta} + H_{\alpha}H_{\beta}\right) + 2j_{(\alpha}n_{\beta)} \tag{22}$$

where $E^2 = E_{\mu}E^{\mu}, H^2 = H_{\mu}H^{\mu}$ and j_{α} is the Poynting vector that measures the energy flow in the spacetime

$$j_{\alpha} = \frac{1}{4\pi}\varepsilon_{\alpha\mu\nu}E^{\mu}H^{\nu}. \tag{23}$$

In order to absorb this EMT by the EMT for anisotropic fluid we choose the direction $n^{\alpha} = u^{\alpha}$ and $\chi^{\alpha} = E^{\alpha}/E$. The latter is possible because when spherical symmetry is imposed $H_{\alpha} = 0$ and E_{α} has only a radial spatial component. The would be heat flow term in Eq (22) disappears and we get

$$T_{\alpha\beta}^{E} = 2eu_{\alpha}u_{\beta} + eg_{\alpha\beta} - 2e\chi_{\alpha}\chi_{\beta}, \quad e = \frac{E^2}{8\pi}. \tag{24}$$

Thus the addition of electric field induces effective pressures and energy density

$$\mu^E = p_t^E = -p_r^E = e \tag{25}$$

related by simple linear equations of state. Hence, a charged perfect fluid or a charged anisotropic fluid may be represented effectively by some neutral anisotropic fluid. There is no heat flow induction in this case.

V. NULL FLUID

Null fluid describes dissipation in the free streaming approximation [4] and adds to the basic EMT the piece

$$T^N_{\alpha\beta} = \varepsilon l_\alpha l_\beta \tag{26}$$

where l^α is the null vector

$$l^\alpha = A^{-1}\delta^\alpha_0 + B^{-1}\delta^\alpha_1 = u^\alpha + \chi^\alpha, \tag{27}$$

satisfying the relations

$$l^\mu l_\mu = 0, \quad l^\mu u_\mu = -1. \tag{28}$$

Substituting Eq (27) into Eq (26) one finds

$$T^N_{\alpha\beta} = \varepsilon u_\alpha u_\beta + \varepsilon\chi_\alpha\chi_\beta + \varepsilon\left(u_\alpha\chi_\beta + u_\beta\chi_\alpha\right). \tag{29}$$

A comparison of this expression with Eq (3), taking into account that $q^\alpha = q\chi^\alpha$, shows that the addition of null fluid generates effective energy density, radial pressure and heat flow, all of them equal

$$\mu^N = p^N_r = q^N = \varepsilon. \tag{30}$$

No tangential pressure is generated. This is the only case where an effective heat flow is induced.

On the other hand, it is known [9], [11] that a perfect fluid plus a null fluid are equivalent to an anisotropic fluid without a heat flow when a rotation of u^α, l^α is done. This is true in any metric. Can we do the same for the sum of anisotropic and null fluid? The Letelier method is based on the condition $\tilde{u}^\mu \tilde{l}_\mu = 0$, imposed on the rotated vectors. Then one of them is necessarily timelike, the other spacelike and the EMT of the sum may be written as in Eq (3) with $q = 0$. However, when the basic model is anisotropic, there is already a spacelike vector, namely χ^α. Therefore we shall rotate l^α into it. Following Ref. [9] we change the signature of the metric and extract from $T_{\alpha\beta} + T^N_{\alpha\beta}$ the part

$$\left(\mu + p_t\right)u_\alpha u_\beta + \varepsilon l_\alpha l_\beta \tag{31}$$

where now $u_\mu u^\mu = 1, \chi^\mu \chi_\mu = -1$. The above expression remains invariant under a rotation with angle α

$$\tilde{u}^\alpha = \cos\alpha\, u^\alpha + C\sin\alpha\, l^\alpha, \quad \tilde{l}^\alpha = -C^{-1}\sin\alpha\, u^\alpha + \cos\alpha\, l^\alpha, \tag{32}$$

$$C = \left(\frac{\varepsilon}{\mu + p_t}\right)^{1/2}. \tag{33}$$

Let us demand that $\tilde{l}^\alpha$ is proportional to χ^α and thus is spacelike

$$\tilde{l}^\alpha = a\chi^\alpha, \quad \tilde{l}^\alpha \tilde{l}_\alpha = -a^2. \tag{34}$$

The relation $\chi^\mu u_\mu = 0$ yields

$$\tan\alpha = Al^\mu u_\mu, \quad a^2 = \frac{\sin^2\alpha}{A^2}. \tag{35}$$

Next we find

$$\tilde{u}^\mu \tilde{u}_\mu = 1 + \sin^2\alpha, \quad \tilde{u}^\mu \tilde{l}_\mu = -\frac{\sin^2\alpha}{A}u^\mu l_\mu. \tag{36}$$

Thus $\tilde{u}^\alpha$ is timelike but is not orthogonal to the spacelike $\tilde{l}^\alpha$. The EMT of the two fluids sum reads

$$(\mu + p_t)\,\tilde{u}_\alpha\tilde{u}_\beta - p_t g_{\alpha\beta} + \left(\frac{p_r - p_t}{a^2} + \varepsilon\right)\tilde{l}_\alpha\tilde{l}_\beta. \tag{37}$$

Now we normalize $\tilde{u}^\alpha, \tilde{l}^\alpha$ and then perform another Letelier rotation to make them orthogonal, $\hat{u}^\mu \hat{l}_\mu = 0$. This can be done when $p_r - p_t + \varepsilon a^2 > 0$. This holds e.g. when $p_r > p_t$. As a result $\hat{u}^\alpha$ stays timelike, while $\hat{l}^\alpha$ stays spacelike and the EMT becomes that of anisotropic fluid without a heat flow. The expressions for the effective energy density and pressures are more complicated than the lengthy formulas in [9] because two rotations have been performed, so we omit them here.

VI.   SUMMARY AND CONCLUSIONS

The results in the previous sections show that viscosity, electric charge and null fluids are equivalent to induced energy density and pressures, related by simple linear equations of state $\mu = np_r$, $n = 0, \pm 1$ and $p_t = kp_r$, $k = 0, \pm 1, -1/2$. When all such additions are combined and absorbed one obtains an effective anisotropic fluid model with

$$\mu^e = \mu + e + \varepsilon, \tag{38}$$

$$p_r^e = p_r - \zeta\Theta - \frac{4}{3}\eta\sigma - e + \varepsilon, \tag{39}$$

$$p_t^e = p_t - \zeta\Theta + \frac{2}{3}\eta\sigma + e + \varepsilon, \tag{40}$$

$$q^e = q + \varepsilon. \tag{41}$$

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These should be plugged into Eq (6) and hence in the l.h.s. of the Einstein equations (1). There are 4 equations for 8 functions $(\mu, p_r, p_t, q, \zeta, \eta, e, \varepsilon)$. The quantities σ, Θ are expressed through the metric. One has to impose 4 relations on these functions or set some of them to zero in order to obtain a determined system of equations. Several conclusions can be drawn.

Only the functions of the two modes of dissipation of energy (q, ε) have effect upon the heat flow.

Viscosity (ζ, η) does not induce effective energy density.

An important characteristic is the anisotropy factor

$$\triangle p^e = p_r^e - p_t^e = \triangle p - 2\eta\sigma - 2e. \tag{42}$$

We see that shear viscosity and charge induce pressure anisotropy. Their absorption by a perfect fluid $(\triangle p = 0)$ makes the latter anisotropic and adds to more sources of anisotropy to the usual ones [1].

Bulk viscosity and null fluid don't induce anisotropy and may be absorbed into an effective perfect fluid model with heat flow.

Charged perfect fluids are related to anisotropic fluids by

$$\triangle p = -2e. \tag{43}$$

In addition to the Einstein equations charged fluids have a Maxwell equation but it serves as a definition of the charge Q in terms of e. Thus results about anisotropic fluids may be carried over to charged perfect fluids. For example, in Ref [12] all static spherically symmetric anisotropic fluid solutions are generated by two arbitrary functions, Z and $\triangle p$. When $\triangle p = 0$ their formula reduces to the result for perfect fluids [13]. When Eq (43) holds instead, we obtain a description of charged perfect fluids in terms of Z and e.

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